

LIE SUPERALGEBRAS AND IRREDUCIBILITY OF $A_1^{(1)}$ -MODULES AT THE CRITICAL LEVEL

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ABSTRACT. We introduce the infinite-dimensional Lie superalgebra \mathcal{A} and construct a family of mappings from certain category of \mathcal{A} -modules to the category of $A_1^{(1)}$ -modules of critical level. Using this approach, we prove the irreducibility of a large family of $A_1^{(1)}$ -modules at the critical level parameterized by $\chi(z) \in \mathbb{C}((z))$. As a consequence, we present a new proof of irreducibility of certain Wakimoto modules. We also give a natural realizations of irreducible quotients of relaxed Verma modules and calculate characters of these representations.

1. INTRODUCTION

In the analysis of certain Fock space representations of infinite-dimensional Lie (super)algebras, one of the main problem is to prove irreducibility of these representations. Irreducible highest weight representations of affine Lie algebras of critical level can be realized by using certain bosonic Fock representation, called the Wakimoto modules (cf. [W], [FF], [FB], [F], [S]). Irreducibility of certain Wakimoto modules gave a very natural proof of the Kac-Kazhdan conjecture on characters of irreducible representations of critical level (cf. [KK]). On the other hand, the category of representations of critical level is much richer than the category \mathcal{O} . So one can investigate the modules outside the category \mathcal{O} and try to understand their structure.

In particular one can investigate the relaxed Verma modules, their irreducible quotients and the corresponding characters. Such kind of representations appeared in the context of representation theory of the affine Lie algebra \widehat{sl}_2 on non-critical levels (cf. [FST], [AM]). In the present paper we shall demonstrate that these relaxed representations appear naturally at the critical level and should be included in the representation theory at this level. We will give a free field realization of the irreducible quotients of relaxed Verma modules of critical level and calculate their characters in the case of affine Lie algebra \widehat{sl}_2 .

Outside the critical level, the representation theory of affine Lie algebra $A_1^{(1)}$ is related to the representation theory of the $N = 2$ superconformal algebra. In [FST],

2000 *Mathematics Subject Classification.* Primary 17B69, Secondary 17B67, 17B68, 81R10.

Key words and phrases. vertex superalgebras, affine Lie algebras, Clifford algebra, Weyl algebra, lattice vertex algebras, critical level.

Partially supported by the MZOS grant 0037125 of the Republic of Croatia.

the authors constructed mappings between certain categories of representations of $\widehat{sl_2}$ and $N = 2$ superconformal algebra. In the context of vertex algebras these mappings was considered in [A1]. But at the critical level the representation theory of $\widehat{sl_2}$ is very different to the representation theory outside the critical level. In particular, the associated vertex algebra $N(-2\Lambda_0)$ contains an infinite-dimensional center (cf. [F]).

In the present paper we find an infinite-dimensional Lie superalgebra \mathcal{A} with the important property that its representation theory is related to those of $\widehat{sl_2}$ at the critical level. This algebra has generators $G^\pm(r)$, $T(n)$, $S(n)$, $r \in \frac{1}{2} + \mathbb{Z}$, $n \in \mathbb{Z}$, which satisfy the following relations

$$\begin{aligned} [S(n), \mathcal{A}] &= [T(n), \mathcal{A}] = 0 \\ \{G^+(r), G^-(s)\} &= 2S(r+s) + (r-s)T(r+s) - (r^2 - \frac{1}{4})\delta_{r+s,0} \\ \{G^+(r), G^+(s)\} &= \{G^-(r), G^-(s)\} = 0 \end{aligned}$$

for all $n \in \mathbb{Z}$, $r, s \in \frac{1}{2} + \mathbb{Z}$.

The main difference between our algebra \mathcal{A} and the $N = 2$ superconformal algebra is in the fact that \mathcal{A} contains large center and that it doesn't contain the Virasoro and Heisenberg subalgebra. Next we consider the vertex superalgebra \mathcal{V} associated to a vacuum representation for \mathcal{A} . This vertex superalgebra is introduced in Section 4 as a vertex subalgebra of $F \otimes M(0)$, where F is a Clifford vertex superalgebra and $M(0)$ a commutative vertex algebra. Then following [A1] we show that there a non-trivial vertex algebra homomorphism $g : N(-2\Lambda_0) \rightarrow \mathcal{V} \otimes F_{-1}$, where F_{-1} is a lattice vertex superalgebra associated to the lattice $\mathbb{Z}\beta$, $\langle \beta, \beta \rangle = -1$. This result allows us to construct $N(-2\Lambda_0)$ -modules from \mathcal{V} -modules. Moreover, we prove that if U is an irreducible \mathcal{V} -module satisfying certain grading condition, then $U \otimes F_{-1} = \bigoplus_{s \in \mathbb{Z}} \mathcal{L}_s(U)$ is a completely reducible $N(-2\Lambda_0)$ -module. Therefore every component $\mathcal{L}_s(U)$ is an irreducible $A_1^{(1)}$ -module at the critical level. It is important to notice that the irreducibility result is proved by using the theory of vertex algebras (cf. Lemma 6.1).

In this way the problem of constructing irreducible $A_1^{(1)}$ -modules is reduced to the construction of irreducible \mathcal{V} -modules. But on irreducible \mathcal{V} -modules, the action of the Lie superalgebra \mathcal{A} can be expressed by the action of generators of infinite-dimensional CL Clifford algebras. By using this fact, in Section 5 we prove the irreducibility of a large family of \mathcal{V} -modules. These modules are parameterized by $\chi(z) \in \mathbb{C}((z))$. In Section 6 we construct mappings \mathcal{L}_s which send irreducible \mathcal{V} -modules to the irreducible $\widehat{sl_2}$ modules at the critical level. As an application, in Section 7 we present a proof of irreducibility of a large family of the Wakimoto modules. In Section 8 we study the irreducible highest weight $\widehat{sl_2}$ -modules of critical level. In particular, we study the simple vertex algebra $L(-2\Lambda_0)$. In Section 9 we get realization of irreducible quotients of relaxed Verma modules. It turns out that these irreducible modules can be realized on certain lattice type vertex algebra.

2. VERTEX ALGEBRA $N(k\Lambda_0)$

We make the assumption that the reader is familiar with the axiomatic theory of vertex superalgebras and their representations (cf. [DL], [FHL], [FLM], [LL], [K2], [Z]).

In this section we recall some basic facts about vertex algebras associated to affine Lie algebras (cf. [FZ], [Li1], [MP]).

Let \mathfrak{g} be a finite-dimensional simple Lie algebra over \mathbb{C} and let (\cdot, \cdot) be a non-degenerate symmetric bilinear form on \mathfrak{g} . Let $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$ be a triangular decomposition for \mathfrak{g} . The affine Lie algebra $\hat{\mathfrak{g}}$ associated with \mathfrak{g} is defined as $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ where c is the canonical central element [K1] and the Lie algebra structure is given by

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + n(x, y)\delta_{n+m,0}c.$$

We will write $x(n)$ for $x \otimes t^n$.

The Cartan subalgebra $\hat{\mathfrak{h}}$ and subalgebras $\hat{\mathfrak{g}}_+$, $\hat{\mathfrak{g}}_-$ of $\hat{\mathfrak{g}}$ are defined by

$$\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c, \quad \hat{\mathfrak{g}}_{\pm} = \mathfrak{g} \otimes t^{\pm 1}\mathbb{C}[t^{\pm 1}].$$

Let $P = \mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}c$ be upper parabolic subalgebra. For every $k \in \mathbb{C}$, let $\mathbb{C}v_k$ be 1-dimensional P -module such that the subalgebra $\mathfrak{g} \otimes \mathbb{C}[t]$ acts trivially, and the central element c acts as multiplication with $k \in \mathbb{C}$. Define the generalized Verma module $N(k\Lambda_0)$ as

$$N(k\Lambda_0) = U(\hat{\mathfrak{g}}) \otimes_{U(P)} \mathbb{C}v_k.$$

Then $N(k\Lambda_0)$ has a natural structure of a vertex algebra. The vacuum vector is $\mathbf{1} = 1 \otimes v_k$.

The vertex algebra $N(k\Lambda_0)$ has very rich representation theory. Let U be any \mathfrak{g} -module. Then U can be consider as a P -module. The induced $\hat{\mathfrak{g}}$ -module $N(k, U) = U(\hat{\mathfrak{g}}) \otimes_{U(P)} U$ is a module for the vertex algebra $N(k\Lambda_0)$.

Let $N^1(k\Lambda_0)$ be the maximal ideal in the vertex algebra $N(k\Lambda_0)$. Then $L(k\Lambda_0) = \frac{N(k\Lambda_0)}{N^1(k\Lambda_0)}$ is a simple vertex algebra.

Let now $\mathfrak{g} = sl_2(\mathbb{C})$ with generators e, f, h and relations $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. Let Λ_0, Λ_1 be the fundamental weights for $\hat{\mathfrak{g}}$.

For $s \in \mathbb{Z}$, we define $H^s = -s\frac{h}{2}$. Then $(H^s, h) = -s$.

Define

$$\Delta_s(z) = z^{H^s(0)} \exp \left(\sum_{n=1}^{\infty} \frac{H^s(n)}{-n} (-z)^{-n} \right).$$

Applying the results obtained in [Li2] on $N(k\Lambda_0)$ -modules we get the following proposition.

Proposition 2.1. *Let $s \in \mathbb{Z}$. For any $N(k\Lambda_0)$ -module $(M, Y_M(\cdot, z))$,*

$$(\pi_s(M), Y_M^s(\cdot, z)) := (M, Y_M(\Delta(H^s, z)\cdot, z))$$

is a $N(k\Lambda_0)$ -module. $\pi_s(M)$ is an irreducible weak $N(k\Lambda_0)$ -module if and only if M is an irreducible weak $N(k\Lambda_0)$ -module.

By definition we have:

$$\begin{aligned}\Delta(H^s, z)e(-1)\mathbf{1} &= z^{-s}e(-1)\mathbf{1}, \\ \Delta(H^s, z)f(-1)\mathbf{1} &= z^sf(-1)\mathbf{1}, \\ \Delta(H^s, z)h(-1)\mathbf{1} &= h(-1)\mathbf{1} - skz^{-1}\mathbf{1}.\end{aligned}$$

In other words, the corresponding automorphism π_s of $U(\hat{\mathfrak{g}})$ satisfies the condition:

$$\pi_s(e(n)) = e(n-s), \quad \pi_s(f(n)) = f(n+s), \quad \pi_s(h(n)) = h(n) - sk\delta_{n,0}.$$

In the case $s = -1$, one can see that

$$\pi_{-1}(L((k-n)\Lambda_0 + n\Lambda_1)) = L(n\Lambda_0 + (k-n)\Lambda_1), \quad \text{for every } n \in \mathbb{Z}_{\geq 0}.$$

It is also important to notice the following important property:

$$\pi_{s+t}(M) \cong \pi_s(\pi_t(M)), \quad (s, t \in \mathbb{Z}).$$

In particular,

$$M \cong \pi_0(M) \cong \pi_s(\pi_{-s}(M)).$$

Lemma 2.1. *Let $x \in \mathbb{C}$. Assume that U is an irreducible $N(k\Lambda_0)$ -module which is generated by the vector v_s ($s \in \mathbb{Z}$) such that:*

$$(2.1) \quad e(n-s)v_s = f(n+s+1)v_s = 0 \quad (n \geq 0),$$

$$(2.2) \quad h(n)v_s = \delta_{n,0}(x+ks)v_s \quad (n \geq 0).$$

Then

$$U \cong \pi_{-s}(L((k-x)\Lambda_0 + x\Lambda_1)).$$

Proof. We consider the $N(k\Lambda_0)$ -module $\pi_s(U)$. By construction, we have that $\pi_s(U)$ is an irreducible highest weight $\hat{\mathfrak{g}}$ -module with the highest weight $(k-x)\Lambda_0 + x\Lambda_1$. Therefore, $\pi_s(U) \cong L((k-x)\Lambda_0 + x\Lambda_1)$, which implies that

$$U \cong \pi_{-s}(\pi_s(U)) \cong \pi_{-s}(L((k-x)\Lambda_0 + x\Lambda_1)),$$

and the Lemma holds. \square

3. CLIFFORD VERTEX SUPERALGEBRAS

The Clifford algebra CL is a complex associative algebra generated by

$$\Psi^\pm(r), \quad r \in \frac{1}{2} + \mathbb{Z},$$

and relations

$$\{\Psi^\pm(r), \Psi^\mp(s)\} = \delta_{r+s,0}; \quad \{\Psi^\pm(r), \Psi^\pm(s)\} = 0$$

where $r, s \in \frac{1}{2} + \mathbb{Z}$.

Let F be the irreducible CL -module generated by the cyclic vector $\mathbf{1}$ such that

$$\Psi^\pm(r)\mathbf{1} = 0 \quad \text{for } r > 0.$$

A basis of F is given by

$$\Psi^+(-n_1 - \frac{1}{2}) \cdots \Psi^+(-n_r - \frac{1}{2}) \Psi^-(-k_1 - \frac{1}{2}) \cdots \Psi^-(-k_s - \frac{1}{2}) \mathbf{1}$$

where $n_i, k_i \in \mathbb{Z}_{\geq 0}$, $n_1 > n_2 > \cdots > n_r$, $k_1 > k_2 > \cdots > k_s$.

Define the following fields on F

$$\Psi^+(z) = \sum_{n \in \mathbb{Z}} \Psi^+(n + \frac{1}{2}) z^{-n-1}, \quad \Psi^-(z) = \sum_{n \in \mathbb{Z}} \Psi^-(n + \frac{1}{2}) z^{-n-1}.$$

The fields $\Psi^+(z)$ and $\Psi^-(z)$ generate on F the unique structure of a simple vertex superalgebra (cf. [Li1], [K2], [FB]).

Define the following Virasoro vector in F :

$$\omega^{(f)} = \frac{1}{2} (\Psi^+(-\frac{3}{2}) \Psi^-(-\frac{1}{2}) + \Psi^-(-\frac{3}{2}) \Psi^+(-\frac{1}{2})) \mathbf{1}.$$

Then the components of the field $L^{(f)}(z) = Y(\omega^{(f)}, z) = \sum_{n \in \mathbb{Z}} L^{(f)}(n) z^{-n-2}$ defines on F a representation of the Virasoro algebra with central charge $c^{(f)} = 1$.

Set

$$J^{(f)}(z) = Y(\Psi^+(-\frac{1}{2}) \Psi^-(-\frac{1}{2}) \mathbf{1}, z) = \sum_{n \in \mathbb{Z}} J^{(f)}(n) z^{-n-1}.$$

Then we have

$$[J^{(f)}(n), \Psi^\pm(m + \frac{1}{2})] = \pm \Psi^\pm(m + n + \frac{1}{2}).$$

Let $\tilde{F} = \text{Ker}_F \Psi^-(\frac{1}{2})$ be the subalgebra of the vertex superalgebra F generated by the fields

$$\partial \Psi^+(z) = \sum_{n \in \mathbb{Z}} -n \Psi^+(n - \frac{1}{2}) z^{-n-1} \quad \text{and} \quad \Psi^-(z) = \sum_{n \in \mathbb{Z}} \Psi^-(n + \frac{1}{2}) z^{-n-1}.$$

Then \tilde{F} is a simple vertex superalgebra with basis

$$(3.3) \quad \Psi^+(-n_1 - \frac{1}{2}) \cdots \Psi^+(-n_r - \frac{1}{2}) \Psi^-(-k_1 - \frac{1}{2}) \cdots \Psi^-(-k_s - \frac{1}{2}) \mathbf{1}$$

where $n_i, k_i \in \mathbb{Z}_{\geq 0}$, $n_1 > n_2 > \cdots > n_r \geq 1$, $k_1 > k_2 > \cdots > k_s \geq 0$.

Let $\overline{F} = \text{Ker}_F \Psi^-(\frac{1}{2}) \cap \text{Ker}_F \Psi^+(\frac{1}{2})$ be the subalgebra of the vertex superalgebra F generated by the fields

$$\partial \Psi^+(z) = \sum_{n \in \mathbb{Z}} -n \Psi^+(n - \frac{1}{2}) z^{-n-1} \quad \text{and} \quad \partial \Psi^-(z) = \sum_{n \in \mathbb{Z}} -n \Psi^-(n - \frac{1}{2}) z^{-n-1}.$$

Then \overline{F} is a simple vertex superalgebra with basis

$$(3.4) \quad \Psi^+(-n_1 - \frac{3}{2}) \cdots \Psi^+(-n_r - \frac{3}{2}) \Psi^-(-k_1 - \frac{3}{2}) \cdots \Psi^-(-k_s - \frac{3}{2}) \mathbf{1}$$

where $n_i, k_i \in \mathbb{Z}_{\geq 0}$, $n_1 > n_2 > \cdots > n_r$, $k_1 > k_2 > \cdots > k_s$.

4. THE VERTEX SUPERALGEBRA \mathcal{V}

In this section we shall define the vertex superalgebra \mathcal{V} and study its representation theory. The vertex superalgebra \mathcal{V} contains a large center. Moreover, the vertex superalgebra \overline{F} is a simple quotient of \mathcal{V} .

Let $M(0) = \mathbb{C}[\gamma^+(-n), \gamma^-(n) \mid n < 0]$ be the commutative vertex algebra generated by the fields

$$\gamma^\pm(z) = \sum_{n < 0} \gamma^\pm(n) z^{-n-1}.$$

(cf. [F]). Let $\chi^\pm(z) = \sum_{n \in \mathbb{Z}} \chi_n^\pm z^{-n-1} \in \mathbb{C}((z))$. Let $M(0, \chi^+, \chi^-)$ denotes the 1-dimensional irreducible $M(0)$ -module with the property that every element $\gamma^\pm(n)$ acts on $M(0, \chi^+, \chi^-)$ as multiplication with $\chi_n^\pm \in \mathbb{C}$.

Let now \mathcal{F} be the vertex superalgebra generated by the fields $\Psi^\pm(z)$ and $\gamma^\pm(z)$. Therefore $\mathcal{F} = F \otimes M(0)$. Denote by \mathcal{V} the vertex subalgebra of the vertex superalgebra \mathcal{F} generated by the following vectors

$$(4.5) \quad \tau^\pm = (\Psi^\pm(-\tfrac{3}{2}) + \gamma^\pm(-1)\Psi^\pm(-\tfrac{1}{2}))\mathbf{1},$$

$$(4.6) \quad j = \frac{\gamma^+(-1) - \gamma^-(-1)}{2}\mathbf{1},$$

$$(4.7) \quad \nu = \frac{2\gamma^+(-1)\gamma^-(-1) + \gamma^+(-2) + \gamma^-(-2)}{4}\mathbf{1}.$$

Then the vertex superalgebra structure on \mathcal{V} is generated by the following fields

$$G^\pm(z) = Y(\tau^\pm, z) = \sum_{n \in \mathbb{Z}} G^\pm(n + \tfrac{1}{2}) z^{-n-2},$$

$$S(z) = Y(\nu, z) = \sum_{n \in \mathbb{Z}} S(n) z^{-n-2},$$

$$T(z) = Y(j, z) = \sum_{n \in \mathbb{Z}} T(n) z^{-n-1}.$$

By using commutator formulae, we have that the components of these fields span an infinite-dimensional Lie superalgebra. Let us denote this Lie superalgebra by \mathcal{A} . This algebra has generators $G^\pm(r)$, $T(n)$, $S(n)$, $r \in \tfrac{1}{2} + \mathbb{Z}$, $n \in \mathbb{Z}$, which satisfy the following relations

$$[S(n), \mathcal{A}] = [T(n), \mathcal{A}] = 0,$$

$$\{G^+(r), G^-(s)\} = 2S(r+s) + (r-s)T(r+s) - (r^2 - \tfrac{1}{4})\delta_{r+s,0},$$

$$\{G^+(r), G^+(s)\} = \{G^-(r), G^-(s)\} = 0$$

for all $n \in \mathbb{Z}$, $r, s \in \tfrac{1}{2} + \mathbb{Z}$.

So the vertex superalgebra \mathcal{V} is generated by the Lie superalgebra \mathcal{A} . Thus we can study \mathcal{V} -modules as modules for the Lie superalgebra \mathcal{A} . The proof of the following proposition is standard.

Proposition 4.1. *We have:*

- (1) $\mathcal{V} = U(\mathcal{A}).\mathbf{1}$

(2) Assume that U is a \mathcal{V} -module. Then U is an irreducible \mathcal{V} -module if and only if U is an irreducible \mathcal{A} -module.

Let \mathcal{V}^{com} be the vertex subalgebra of \mathcal{V} generated by the fields $S(z)$ and $T(z)$. \mathcal{V}^{com} is a commutative vertex algebra.

The operator $J^f(0)$ acts semisimply on the vertex superalgebra \mathcal{V} and defines the following \mathbb{Z} -graduation:

$$\begin{aligned}
 \mathcal{V} &= \bigoplus_{m \in \mathbb{Z}} \mathcal{V}^m, \quad \text{where} \\
 \mathcal{V}^m &= \{v \in \mathcal{V} \mid J^f(0)v = mv\} \\
 &= \text{span}_{\mathbb{C}}\{G^+(-n_1 - \frac{3}{2}) \cdots G^+(-n_r - \frac{3}{2})G^-(-k_1 - \frac{3}{2}) \cdots G^-(-k_s - \frac{3}{2})w \mid \\
 (4.8) \quad &w \in \mathcal{V}^{com}, n_i, k_j \in \mathbb{Z}_{\geq 0}, r - s = m\}.
 \end{aligned}$$

It is clear that $\mathcal{V}^{com} \subset \mathcal{V}^0$.

Every $U(\mathcal{A})$ -submodule of \mathcal{V} becomes an ideal in the vertex superalgebra \mathcal{V} . Let $I^{com} = U(\mathcal{A}).\mathcal{V}^{com}$ be the ideal in \mathcal{V} generated by \mathcal{V}^{com} . From the definition of vertex superalgebras \mathcal{V} and \overline{F} we get the following result.

Proposition 4.2. *The quotient vertex superalgebra \mathcal{V}/I^{com} is isomorphic to the simple vertex superalgebra \overline{F} .*

5. IRREDUCIBILITY OF CERTAIN \mathcal{V} -MODULES

In this section we shall consider a family of irreducible \mathcal{V} -modules.

For $\chi^+, \chi^- \in \mathbb{C}((z))$ we set $F(\chi^+, \chi^-) := F \otimes M(0, \chi^+, \chi^-)$.

Then $F(\chi^+, \chi^-)$ is a module for the vertex superalgebra \mathcal{V} , and therefore for the Lie superalgebra \mathcal{A} .

Since $M(0, \chi^+, \chi^-)$ is one-dimensional, we have that as a vector space

$$(5.9) \quad F(\chi^+, \chi^-) \cong F \cong \bigwedge (\Psi^\pm(-i - \frac{1}{2}) \mid i \geq 0).$$

This actually shows that for every $\chi^+, \chi^- \in \mathbb{C}((z))$ on the vertex superalgebra F exists the structure of a \mathcal{A} -structure. In this section we shall use this identification.

Proposition 5.1. *Assume that $\lambda \in \mathbb{C} \setminus \mathbb{Z}$, $p \in \mathbb{Z}_{\geq 0}$ and that*

$$\chi(z) = \sum_{n=-p}^{\infty} \chi_{-n} z^{n-1} \in \mathbb{C}((z))$$

satisfies the following conditions

$$(5.10) \quad \chi_p \neq 0,$$

$$(5.11) \quad \chi_0 \in \mathbb{C} \setminus \mathbb{Z} \quad \text{if } p = 0.$$

Then $F(\frac{\lambda}{z}, \chi)$ is an irreducible \mathcal{V} -module.

Proof. Since $F(\frac{\lambda}{z}, \chi)$ is a \mathcal{V} -module, it remains to prove that it is an irreducible module for the Lie superalgebra \mathcal{A} . Recall (5.9). The \mathcal{A} -module structure on $F(\frac{\lambda}{z}, \chi)$ is uniquely determined by the following action of generators of \mathcal{A} on F :

$$(5.12) \quad G^+(i - \frac{1}{2}) = (\lambda - i)\Psi^+(i - \frac{1}{2}),$$

$$(5.13) \quad G^-(i - \frac{1}{2}) = -i\Psi^-(i - \frac{1}{2}) + \sum_{k=-p}^{\infty} \chi_{-k}\Psi^-(k + i - \frac{1}{2}).$$

First we shall prove that the vacuum vector is a cyclic vector of the $U(\mathcal{A})$ -action, i.e.,

$$(5.14) \quad U(\mathcal{A}).\mathbf{1} = F.$$

Take an arbitrary basis element

$$(5.15) \quad v = \Psi^+(-n_1 - \frac{1}{2}) \cdots \Psi^+(-n_r - \frac{1}{2}) \Psi^-(-k_1 - \frac{1}{2}) \cdots \Psi^-(-k_s - \frac{1}{2}) \mathbf{1} \in F,$$

where $n_i, k_i \in \mathbb{Z}_{\geq 0}$, $n_1 > n_2 > \cdots > n_r \geq 0$, $k_1 > k_2 > \cdots > k_s \geq 0$.

Let $N \in \mathbb{Z}_{\geq 0}$ such that $N \geq k_1$. By using (5.13) we get that

$$G^-(p - N - \frac{1}{2}) \cdots G^-(p - \frac{3}{2}) G^-(p - \frac{1}{2}) \mathbf{1} = C \Psi^-(-N - \frac{1}{2}) \cdots \Psi^-(-\frac{3}{2}) \Psi^-(-\frac{1}{2}) \mathbf{1},$$

where

$$C = \begin{cases} \chi_p^{N+1} & \text{if } p \geq 1 \\ \chi_0(\chi_0 + 1) \cdots (\chi_0 + N) & \text{if } p = 0 \end{cases}.$$

So $C \neq 0$, and we have that

$$\Psi^-(-N - \frac{1}{2}) \cdots \Psi^-(-\frac{3}{2}) \Psi^-(-\frac{1}{2}) \mathbf{1} \in U(\mathcal{A}).\mathbf{1}.$$

By using this fact and the action of elements $G^+(i - \frac{1}{2})$, $i \in \mathbb{Z}$, we obtain that $v \in U(\mathcal{A}).\mathbf{1}$. In this way we proved (5.14).

In order to prove irreducibility, it is enough to show that arbitrary basis element v of the form (5.15) is cyclic in F . This follows from (5.14) and the fact that

$$G^-(n_r + p + \frac{1}{2}) \cdots G^-(n_1 + p + \frac{1}{2}) G^+(k_s + \frac{1}{2}) \cdots G^+(k_1 + \frac{1}{2}) v = C' \mathbf{1},$$

where the non-trivial constant C' is given by

$$C' = (-1)^{r+s} (\lambda - k_1 - 1) \cdots (\lambda - k_s - 1) \cdot C'' \quad \text{and} \\ C'' = \begin{cases} \chi_p^r & \text{if } p \geq 1 \\ (\chi_0 - n_1 - 1) \cdots (\chi_0 - n_r - 1) & \text{if } p = 0 \end{cases}.$$

□

This proposition has the following important consequence.

Corollary 5.1. *Assume that $\lambda, \mu \in \mathbb{C} \setminus \mathbb{Z}$. Then $F(\frac{\lambda}{z}, \frac{\mu}{z})$ is an irreducible \mathcal{V} -module.*

Now let $\chi(z) \in \mathbb{C}((z))$. Define :

$$\tilde{F}_\chi := \tilde{F} \otimes M(0, 0, \chi).$$

It is clear that \tilde{F}_χ is a submodule of the \mathcal{V} -module $F(0, \chi)$. Now we shall prove the following important irreducibility result:

Proposition 5.2. Assume that $p \in \mathbb{Z}_{\geq 0}$ and that

$$\chi(z) = \sum_{n=-p}^{\infty} \chi_{-n} z^{n-1} \in \mathbb{C}((z))$$

satisfies the following conditions

$$(5.16) \quad \chi_p \neq 0,$$

$$(5.17) \quad \chi_0 \in \{1\} \cup (\mathbb{C} \setminus \mathbb{Z}) \quad \text{if } p = 0.$$

Then \tilde{F}_χ is an irreducible \mathcal{V} -module.

Proof. Since \tilde{F}_χ is a \mathcal{V} -module, it remains to prove that \tilde{F}_χ is an irreducible module for the Lie superalgebra \mathcal{A} . The \mathcal{A} -module structure on \tilde{F}_χ is uniquely determined by the following action of the Lie superalgebra \mathcal{A} on \tilde{F} :

$$(5.18) \quad G^+(i - \tfrac{1}{2}) = -i\Psi^+(i - \tfrac{1}{2}),$$

$$(5.19) \quad G^-(i - \tfrac{1}{2}) = -i\Psi^-(i - \tfrac{1}{2}) + \sum_{k=-p}^{\infty} \chi_{-k} \Psi^-(k + i - \tfrac{1}{2}).$$

By using this action, the basis description (3.3) of \tilde{F} , and same proof to those of Proposition 5.1 we get the irreducibility result. \square

Corollary 5.2. Assume that $\lambda \in \mathbb{C} \setminus \mathbb{Z}$ or $\lambda = -1$. Then $\tilde{F}_{-\frac{\lambda}{z}}$ is an irreducible \mathcal{V} -module.

Proposition 5.3. Assume that $m, n \in \mathbb{Z}_{\geq 0}$. Then

$$\begin{aligned} \overline{F}(-\tfrac{m}{z}, -\tfrac{n}{z}) &= \text{Ker}_{F(-\frac{m}{z}, -\frac{n}{z})} G^-(m + \tfrac{1}{2}) \bigcap \text{Ker}_{F(-\frac{m}{z}, -\frac{n}{z})} G^+(n + \tfrac{1}{2}) \\ &\cong \bigwedge (\Psi^+(-i - \tfrac{1}{2}), \Psi^-(-j - \tfrac{1}{2}) \mid i, j \in \mathbb{Z}_{\geq 0}, i \neq m, j \neq n) \end{aligned}$$

is an irreducible \mathcal{V} -module. In particular, the \mathcal{V} -module $\overline{F}(0, -\frac{n}{z})$ is irreducible.

Proof. First we notice that on $F(-\frac{m}{z}, -\frac{n}{z})$

$$(5.20) \quad G^+(i - \tfrac{1}{2}) = -(i + m)\Psi^+(i - \tfrac{1}{2}), \quad G^-(i - \tfrac{1}{2}) = -(i + n)\Psi^-(i - \tfrac{1}{2}).$$

This implies that

$$(5.21) \quad \overline{F}(-\tfrac{m}{z}, -\tfrac{n}{z}) \cong \bigwedge (\Psi^+(-i - \tfrac{1}{2}), \Psi^-(-j - \tfrac{1}{2}) \mid i, j \in \mathbb{Z}_{\geq 0}, i \neq m, j \neq n)$$

is a \mathcal{V} -submodule of $F(-\frac{m}{z}, -\frac{n}{z})$. It remains to prove that $\overline{F}(-\frac{m}{z}, -\frac{n}{z})$ is an irreducible \mathcal{A} -module. By using (5.20) we have that $\overline{F}(-\frac{m}{z}, -\frac{n}{z})$ has the structure of a module for the subalgebra ${}^{m,n}CL$ of the Clifford algebra CL generated by

$$\Psi^+(-i - \tfrac{1}{2}), \Psi^-(-j - \tfrac{1}{2}), \quad i, j \in \mathbb{Z}, \quad i \neq m, \quad j \neq n.$$

Since $\bigwedge (\Psi^+(-i - \tfrac{1}{2}), \Psi^-(-j - \tfrac{1}{2}) \mid i, j \in \mathbb{Z}_{\geq 0}, i \neq m, j \neq n)$ is an irreducible ${}^{m,n}CL$ -module, we have that $\overline{F}(-\frac{m}{z}, -\frac{n}{z})$ is an irreducible \mathcal{A} -module. \square

6. VERTEX ALGEBRAS AT THE CRITICAL LEVEL

In previous sections we investigated the properties of the vertex superalgebra \mathcal{V} . This vertex superalgebra is generated by the Lie superalgebra \mathcal{A} which is similar to the $N = 2$ superconformal algebras. This makes the vertex superalgebra \mathcal{V} similar to the $N = 2$ vertex superalgebras investigated in [EG], [FST] and [A1]. The main difference is that \mathcal{V} doesn't contain the Virasoro and the Heisenberg subalgebra. On the other hand \mathcal{V} contains large center. One important property of the $N = 2$ vertex superalgebras is their connection to the affine \widehat{sl}_2 -vertex algebras. The most effective way for studying this connection is by using Kazama-Suzuki and anti Kazama-Suzuki mapping(cf. [KS], [FST]). Motivated by the anti Kazama-Suzuki mapping, we shall get a realization of the vertex algebras associated to the \widehat{sl}_2 -modules at the critical level.

Let F_{-1} , be the lattice vertex superalgebra $V_{\mathbb{Z}\beta}$, associated to the lattice $\mathbb{Z}\beta$, where $\langle \beta, \beta \rangle = -1$. As a vector space, F_{-1} is isomorphic to $M_\beta(1) \otimes \mathbb{C}[L]$, where $M_\beta(1)$ is a level one irreducible module for the Heisenberg algebra $\hat{\mathfrak{h}}_{\mathbb{Z}}$ associated to the one dimensional abelian algebra $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$ and $\mathbb{C}[L]$ is the group algebra with a generator e^β . The generators of F_{-1} are e^β and $e^{-\beta}$. Moreover, F_{-1} is a simple vertex superalgebra and a completely reducible $M_\beta(1)$ -module isomorphic to

$$F_{-1} \cong \bigoplus_{m \in \mathbb{Z}} F_{-1}^m,$$

where F_{-1}^m is an irreducible $M_\beta(1)$ -module generated by $e^{m\beta}$.

We shall now consider the vertex superalgebra $\mathcal{V} \otimes F_{-1}$. Let Y be the vertex operator defining the vertex operator superalgebra structure on $\mathcal{V} \otimes F_{-1}$. For every $v \in \mathcal{V} \otimes F_{-1}$, let $Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}$.

Define

$$(6.22) \quad e = G^+(-\frac{3}{2})\mathbf{1} \otimes e^\beta,$$

$$(6.23) \quad h = -2(\mathbf{1} \otimes \beta(-1) - T(-1) \otimes \mathbf{1}),$$

$$(6.24) \quad f = G^-(-\frac{3}{2})\mathbf{1} \otimes e^{-\beta}.$$

For $x \in \text{span}_{\mathbb{C}}\{e, f, h\}$ set $x(z) = Y(x, z) = \sum_{n \in \mathbb{Z}} x(n)z^{-n-1}$. Then the components of the field $e(z)$, $f(z)$ and $h(z)$ satisfy the commutation relations for the affine Lie algebra \widehat{sl}_2 of level -2 . In particular we have

$$\begin{aligned} e(n) &= \sum_{i \in \mathbb{Z}} G^+(i - \frac{1}{2}) \otimes e_{n-i-1}^\beta, \\ h(n) &= -2\beta(n) + 2T(n), \\ f(n) &= \sum_{i \in \mathbb{Z}} G^-(i - \frac{1}{2}) \otimes e_{n-i-1}^{-\beta}. \end{aligned}$$

Denote by V the subalgebra of $\mathcal{V} \otimes F_{-1}$ generated by e , f and h , i.e.,

$$V = \text{span}_{\mathbb{C}}\{u_{n_1}^1 \cdots u_{n_r}^r (\mathbf{1} \otimes \mathbf{1}) \mid u^1, \dots, u^r \in \{e, f, h\}, n_1, \dots, n_r \in \mathbb{Z}, r \in \mathbb{Z}_{\geq 0}\}.$$

As a $U(\hat{\mathfrak{g}})$ -module, V is a cyclic module generated by the vacuum vector $\mathbf{1} \otimes \mathbf{1}$. This implies that V is a certain quotient of the vertex algebra $N(-2\Lambda_0)$.

So we have:

Proposition 6.1. *There exists a non-trivial homomorphism of vertex algebras*

$$g : N(-2\Lambda_0) \rightarrow \mathcal{V} \otimes F_{-1},$$

which is uniquely determined by (6.22)-(6.24).

Define the vector

$$(6.25) \quad H = \Psi^+(-\tfrac{1}{2})\Psi^-(-\tfrac{1}{2})\mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \beta(-1) \in F \otimes F_{-1} \subset \mathcal{F} \otimes F_{-1}.$$

Then the operator $H(0)$ acts semisimply on vertex superalgebras $\mathcal{F} \otimes F_{-1}$ and $\mathcal{V} \otimes F_{-1}$. Let

$$W(s) = \{v \in \mathcal{V} \otimes F_{-1} \mid H(0)v = sv\}.$$

Then we have the following decomposition:

$$\mathcal{V} \otimes F_{-1} \cong \oplus_{s \in \mathbb{Z}} W(s).$$

Let \hat{t} be the (commutative) Lie algebra generated by the components of the field $T(z)$, and let $M_T(0) \cong \mathbb{C}[T(-1), T(-2), \dots]$ be the associated (commutative) vertex algebra.

Let $\hat{\mathfrak{g}}^{ext} = \hat{\mathfrak{g}} \oplus \hat{t}$ be the extension of the affine Lie algebra $\hat{\mathfrak{g}}$ by the Lie algebra \hat{t} such that every element of \hat{t} is in the center of $\hat{\mathfrak{g}}^{ext}$.

Theorem 6.1. *The vertex algebra $W(0)$ is generated by e, f, h, j and*

$$W(0) = U(\hat{\mathfrak{g}}^{ext}).(\mathbf{1} \otimes \mathbf{1}) \cong V \otimes M_T(0).$$

Proof. Let \mathcal{U} be the vertex subalgebra of $\mathcal{V} \otimes F_{-1}$ generated by the set $\{e, f, h, j\}$. The components of the fields $e(z), f(z), h(z), T(z)$ span the Lie algebra $\hat{\mathfrak{g}}^{ext}$. Since the field $T(z)$ commutes with the action of $\hat{\mathfrak{g}}$ we have that $\mathcal{U} = U(\hat{\mathfrak{g}}^{ext}).(\mathbf{1} \otimes \mathbf{1}) \cong V \otimes M_T(0)$.

Since the operator $H(0)$ acts trivially on the vertex algebra \mathcal{U} , we conclude that $\mathcal{U} \subset W(0)$. We shall now prove that the vectors e, f, h, j generate $W(0)$. First we notice that

$$(6.26) \quad W(0) = \oplus_{m \in \mathbb{Z}} \mathcal{V}^m \otimes F_{-1}^m.$$

Since

$$S(-2)\mathbf{1} \otimes \mathbf{1} = \tfrac{1}{4}(e(-1)f(-1) + f(-1)e(-1) + \tfrac{1}{2}h(-1)^2).(\mathbf{1} \otimes \mathbf{1}) - \tfrac{1}{2}T(-1)^2\mathbf{1} \otimes \mathbf{1},$$

we have that $S(-2)\mathbf{1} \otimes \mathbf{1} \in \mathcal{U}$. Therefore,

$$(6.27) \quad \mathcal{V}^{com} \subset \mathcal{U}.$$

We note that the following relations hold:

$$(6.28) \quad G^+(-\frac{3}{2})\mathbf{1} \otimes \mathbf{1} = e(-2).(\mathbf{1} \otimes e^{-\beta}),$$

$$(6.29) \quad [e(m), (\mathbf{1} \otimes e^{-\beta})_n] = 0, \quad \text{for every } m, n \in \mathbb{Z},$$

$$(6.30) \quad G^-(-\frac{3}{2})\mathbf{1} \otimes \mathbf{1} = f(-2).(\mathbf{1} \otimes e^{\beta}),$$

$$(6.31) \quad [f(m), (\mathbf{1} \otimes e^{\beta})_n] = 0, \quad \text{for every } m, n \in \mathbb{Z}.$$

$$(6.32) \quad \mathbf{1} \otimes \beta(-1) = -\frac{1}{2}(h(-1) - 2T(-1)).(\mathbf{1} \otimes \mathbf{1}),$$

Relations (6.27) and (6.32) imply that

$$(6.33) \quad \mathcal{V}^{com} \otimes F_{-1}^0 = \mathcal{V}^{com} \otimes M_{\beta}(1) \subset \mathcal{U}.$$

We shall now prove that $\mathcal{V}^m \otimes F_{-1}^m \subset \mathcal{U}$, where $m \in \mathbb{Z}$. By using the description of \mathcal{V}^m from (4.8), we conclude that it is enough to show that elements of the form

$$\begin{aligned} v &= G^+(-n_1 - \frac{3}{2}) \cdots G^+(-n_r - \frac{3}{2}) G^-(-k_1 - \frac{3}{2}) \cdots G^-(-k_s - \frac{3}{2}) w \otimes w^{(1)} \\ &w \in \mathcal{V}^{com}, w^{(1)} \in F_{-1}^m, n_i, k_{\ell} \in \mathbb{Z}_{\geq 0}, r - s = m, \end{aligned}$$

belong to \mathcal{U} .

By using (6.28) and (6.29) we get that

$$v = \sum_{i \in I} \tilde{g}_i^{(2)} \cdot \left(G^-(-k_1 - \frac{3}{2}) \cdots G^-(-k_s - \frac{3}{2}) w \otimes w_i^{(2)} \right)$$

for certain $w_i^{(2)} \in F_{-1}^{-s}$, $\tilde{g}_i^{(2)} \in U(\mathbb{C}e \otimes \mathbb{C}[t, t^{-1}]) \subset U(\hat{\mathfrak{g}})$ and a finite index set I . Similarly, applying (6.30) and (6.31) we get

$$G^-(-k_1 - \frac{3}{2}) \cdots G^-(-k_s - \frac{3}{2}) w \otimes w_i^{(2)} = \sum_{\ell \in I_i} \tilde{g}_{i,\ell}^{(3)} \cdot (w \otimes w_{i,\ell}^{(3)})$$

for certain $w_{i,\ell}^{(3)} \in F_{-1}^0 = M_{\beta}(1)$, $\tilde{g}_{i,\ell}^{(3)} \in U(\mathbb{C}f \otimes \mathbb{C}[t, t^{-1}]) \subset U(\hat{\mathfrak{g}})$, and a finite index set I_i . Since $w \otimes w_{i,\ell}^{(3)} \in \mathcal{V}^{com} \otimes M_{\beta}(1)$, we obtain

$$v \in U(\hat{\mathfrak{g}}).(\mathcal{V}^{com} \otimes M_{\beta}(1)) \subset U(\hat{\mathfrak{g}}^{ext}).(\mathbf{1} \otimes \mathbf{1}) = \mathcal{U}.$$

Therefore,

$$\mathcal{V}^m \otimes F_{-1}^m \subset \mathcal{U}, \quad \text{for every } m \in \mathbb{Z}.$$

Now (6.26) implies that $W(0) = \mathcal{U}$. \square

The next Lemma follows from Corollary 4.2 of [DM]. This result will be our important tool in the irreducibility analysis.

Lemma 6.1. *Assume that M is an irreducible $\mathcal{V} \otimes F_{-1}$ -module. Then for each $0 \neq w \in M$, M is spanned as a $\mathcal{V} \otimes F_{-1}$ -module by $u_n w$, for $u \in \mathcal{V} \otimes F_{-1}$ and $n \in \mathbb{Z}$.*

Lemma 6.2. *Assume that R is an irreducible $W(0)$ -module. Then R is an irreducible V -module. In particular, R is an irreducible $\hat{\mathfrak{g}}$ -module at the critical level.*

Proof. Since $W(0) \cong V \otimes M_T(0)$, we have that $R \cong S \otimes N$, where S is an irreducible V -module and N is an irreducible $M_T(0)$ -module. Since $M_T(0)$ is a commutative vertex algebra, we have that N is one dimensional and that every element $T(n)$ acts on R as a scalar multiplication. Therefore R is irreducible as a V -module. \square

Theorem 6.2. *Assume that U is a \mathcal{V} -module such that U admits the following \mathbb{Z} -graduation*

$$(6.34) \quad U = \bigoplus_{j \in \mathbb{Z}} U^j, \quad \mathcal{V}^i \cdot U^j \subset U^{i+j}.$$

Then

$$U \otimes F_{-1} = \bigoplus_{s \in \mathbb{Z}} \mathcal{L}_s(U), \quad \text{where } \mathcal{L}_s(U) := \bigoplus_{i \in \mathbb{Z}} U^i \otimes F_{-1}^{-s+i},$$

is an $W(0)$ -module.

If U is irreducible, then for every $s \in \mathbb{Z}$ $\mathcal{L}_s(U)$ is an irreducible V -module.

Proof. Since $\mathcal{V} \otimes F_{-1} = \bigoplus_{\ell \in \mathbb{Z}} W(\ell)$, relation (6.34) implies that

$$(6.35) \quad W(\ell) \cdot \mathcal{L}_s(U) \subset \mathcal{L}_{s+\ell}(U) \text{ for every } \ell, s \in \mathbb{Z}.$$

This proves that $\mathcal{L}_s(U)$ is a $W(0)$ -module for every $s \in \mathbb{Z}$.

Assume now that U is irreducible. Then $U \otimes F_{-1}$ is an irreducible $\mathcal{V} \otimes F_{-1}$ -module. Let $0 \neq v \in \mathcal{L}_s(U)$. Since $U \otimes F_{-1}$ is a simple $\mathcal{V} \otimes F_{-1}$ -module, by Lemma 6.1 we get that

$$(6.36) \quad U \otimes F_{-1} = \text{span}_{\mathbb{C}}\{u_n v \mid u \in \mathcal{V} \otimes F_{-1}, n \in \mathbb{Z}\}.$$

By using (6.35) and (6.36) we conclude that

$$(6.37) \quad \mathcal{L}_s(U) = \text{span}_{\mathbb{C}}\{u_n v \mid u \in W(0), n \in \mathbb{Z}\}.$$

So $\mathcal{L}_s(U)$ is an irreducible $W(0)$ -module. Now Lemma 6.2 gives that $\mathcal{L}_s(U)$ is an irreducible V -module, and therefore an irreducible $\hat{\mathfrak{g}}$ -module of critical level. \square

Corollary 6.1. *Assume that $U \subset F(\chi^+, \chi^-)$ is an irreducible \mathcal{V} -module. Then $U \otimes F_{-1}$ is a completely reducible V -module*

$$U \otimes F_{-1} = \bigoplus_{s \in \mathbb{Z}} \mathcal{L}_s(U)$$

and $\mathcal{L}_s(U) = \{v \in U \otimes F_{-1} \mid H(0)v = sv\}$ is an irreducible V -module. Moreover, $\mathcal{L}_s(U)$ is an irreducible $\hat{\mathfrak{g}}$ -module at the critical level.

Proof. The operator $J^f(0)$ acts semisimply on $U \subset F(\chi^+, \chi^-)$ and defines on U the following graduation

$$U = \bigoplus_{j \in \mathbb{Z}} U^j, \quad \text{where } U^j = \{u \in U \mid J^f(0)u = ju\}.$$

Now Theorem 6.2 implies that $\mathcal{L}_s(U)$ is an irreducible $\hat{\mathfrak{g}}$ -module at the critical level. \square

7. WEYL VERTEX ALGEBRA AND IRREDUCIBILITY OF THE WAKIMOTO MODULES

In this section we will see that our $\hat{\mathfrak{g}}$ -modules include the Wakimoto $\hat{\mathfrak{g}}$ -modules at the critical level defined by using vertex algebra W associated to the Weyl algebra. As an application, we present a proof of irreducibility for a family of Wakimoto modules.

First we shall consider the simple vertex superalgebra $\tilde{F} \otimes F_{-1}$. The operator $H(0)$ acts semisimply on $\tilde{F} \otimes F_{-1}$, and we have that

$$\text{Ker}_{\tilde{F} \otimes F_{-1}} H(0) = \mathcal{L}_0(\tilde{F})$$

is a simple vertex algebra. We shall now identify this vertex algebra.

Define:

$$(7.38) \quad a := \Psi^+(-\frac{3}{2})\mathbf{1} \otimes e^\beta, \quad a^* := -\Psi^-(-\frac{1}{2})\mathbf{1} \otimes e^{-\beta},$$

and

$$a(z) = Y(a, z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}, \quad a^*(z) = Y(a^*, z) = \sum_{n \in \mathbb{Z}} a^*(n) z^{-n}.$$

Then

$$[a(n), a(m)] = [a^*(n), a^*(m)] = 0, \quad [a(n), a^*(m)] = \delta_{n+m, 0}.$$

Therefore, the components of the fields $a(z)$ and $a^*(z)$ span the infinite dimensional Weyl algebra. Let W be the vertex subalgebra of $\tilde{F} \otimes F_{-1}$ generated by $a(z)$ and $a^*(z)$ (cf. [FMS], [F], [A2]).

By using a similar proof to those of Theorem 6.1 we get.

Proposition 7.1. *The vertex algebra $\mathcal{L}_0(\tilde{F})$ is generated by a and a^* . Thus we have:*

$$(7.39) \quad W \cong \mathcal{L}_0(\tilde{F}).$$

For every $\chi \in \mathbb{C}((z))$, the vector space

$$\tilde{F}_\chi \otimes F_{-1} \cong \tilde{F} \otimes F_{-1}$$

carries the structure of a $\hat{\mathfrak{g}}$ -module at the critical level, and $\mathcal{L}_0(\tilde{F}_\chi) \subset \tilde{F} \otimes F_{-1}$ is a $\hat{\mathfrak{g}}$ -submodule. By construction, we have that as a vector space $\mathcal{L}_0(\tilde{F}_\chi)$ is isomorphic to $\mathcal{L}_0(\tilde{F}) \cong W$. Therefore on the vertex algebra W for every $\chi \in \mathbb{C}((z))$ exists a $\hat{\mathfrak{g}}$ -structure.

By using the definition of $\hat{\mathfrak{g}}$ -structure on W and (7.38) one gets:

$$\begin{aligned} e(z) &= a(z), \\ h(z) &= -2 : a^*(z)a(z) : - \chi(z) \\ f(z) &= - : a^*(z)^2 a(z) : - 2\partial_z a^*(z) - a^*(z)\chi(z). \end{aligned}$$

Therefore the $\hat{\mathfrak{g}}$ -module structure on W coincides with the Wakimoto module $W_{-\chi}$ (see [F] and reference therein).

Combining Proposition 5.2 and Corollary 6.1 we obtain the following irreducibility result.

Theorem 7.1.

- (1) For every $\chi \in \mathbb{C}((z))$, the $\hat{\mathfrak{g}}$ -module $\mathcal{L}_0(\tilde{F}_\chi)$ is isomorphic to the Wakimoto module $W_{-\chi}$.
 (2) Assume that

$$\chi(z) = \sum_{n=-p}^{\infty} \chi_{-n} z^{n-1} \in \mathbb{C}((z)) \quad (p \in \mathbb{Z}_{\geq 0})$$

satisfies the conditions (5.16) and (5.17) of Proposition 5.2. Then $W_{-\chi} \cong \mathcal{L}_0(\tilde{F}_\chi)$ is an irreducible $\hat{\mathfrak{g}}$ -module at the critical level.

8. CONSTRUCTION OF IRREDUCIBLE HIGHEST WEIGHT MODULES

In this section we apply the results from previous sections and obtain a construction of all irreducible highest weight $\hat{\mathfrak{g}}$ -modules $L(\lambda)$ of level -2 . It turns out that modules $L(\lambda)$ are realized inside the Weyl vertex algebra W , and therefore they are submodules of certain Wakimoto modules. By using the methods developed in [A3], we shall identify modules obtained from irreducible highest weight modules by applying the automorphism π_s . We will also show the vertex superalgebra $\overline{F} \otimes F_{-1}$ is a completely reducible module for the simple vertex algebra $L(-2\Lambda_0)$.

First we shall study non-generic highest weight representations.

Theorem 8.1.

- (i) For every $n \in \mathbb{Z}_{\geq 0}$ the vector space $\overline{F}(0, -\frac{n}{z}) \otimes F_{-1}$ carries an $\hat{\mathfrak{g}}$ -structure uniquely determined by

$$\begin{aligned} e(m) &= - \sum_{i \in \mathbb{Z}} i \Psi^+(i - \frac{1}{2}) \otimes e_{m-i-1}^\beta \\ f(m) &= - \sum_{i \in \mathbb{Z}} (i+n) \Psi^-(i - \frac{1}{2}) \otimes e_{m-i-1}^{-\beta} \\ h(m) &= -2\beta(m) + n\delta_{m,0} \\ c &= -2, \end{aligned}$$

where $m \in \mathbb{Z}$. Moreover, $\overline{F}(0, -\frac{n}{z}) \otimes F_{-1}$ is a completely reducible $\hat{\mathfrak{g}}$ -module and

$$\overline{F}(0, -\frac{n}{z}) \otimes F_{-1} \cong \bigoplus_{s \in \mathbb{Z}} \pi_s(L(-(2+n)\Lambda_0 + n\Lambda_1)).$$

- (ii)

$$U(\hat{\mathfrak{g}}).(\mathbf{1} \otimes \mathbf{1}) \cong (L(-(2+n)\Lambda_0 + n\Lambda_1), \quad U(\hat{\mathfrak{g}}).(\mathbf{1} \otimes e^{-\beta}) \cong (L(n\Lambda_0 - (n+2)\Lambda_1).$$

Proof. By using Proposition 5.3 we have that for every $n \in \mathbb{Z}_{\geq 0}$, $\overline{F}(0, -\frac{n}{z})$ is an irreducible \mathcal{V} -module. Then Corollary 6.1 gives that $\overline{F}(0, -\frac{n}{z}) \otimes F_{-1}$ is a completely

reducible $\hat{\mathfrak{g}}$ -module isomorphic to $\oplus_{s \in \mathbb{Z}} \mathcal{L}_s(\overline{F}(0, -\frac{n}{z}))$, where $\mathcal{L}_s(\overline{F}(0, -\frac{n}{z}))$ is an irreducible $\hat{\mathfrak{g}}$ -module. For every $s \in \mathbb{Z}$, we set

$$v_s = \mathbf{1} \otimes e^{-s\beta} \in \mathcal{L}_s(\overline{F}(0, -\frac{n}{z})).$$

Now using Lemma 2.1 one obtains that

$$\mathcal{L}_s(\overline{F}(0, -\frac{n}{z})) = U(\hat{\mathfrak{g}}).v_s = \pi_{-s}(L(-(2+n)\Lambda_0 + n\Lambda_1)).$$

This proves (i). The second assertion follows from (i) and from the fact that $\pi_{-1}(L(-(2+n)\Lambda_0 + n\Lambda_1)) \cong L(n\Lambda_0 - (2+n)\Lambda_1)$. \square

When $n \in \mathbb{Z}_{>0}$, then $L(-(2+n)\Lambda_0 + n\Lambda_1)$ is not a module for the simple vertex algebra $L(-2\Lambda_0)$. So Theorem 8.1 can be applied only in the framework of $N(-2\Lambda_0)$ -modules. But when $n = 0$ we have that \mathcal{V} -module $\overline{F} = \overline{F}(0, 0)$ is a simple vertex superalgebra (see Proposition 4.2), and we have the following realization of $L(-2\Lambda_0)$ -modules.

Corollary 8.1. *The simple vertex algebra $L(-2\Lambda_0)$ is a subalgebra of the vertex superalgebra $\overline{F} \otimes F_{-1}$, and we have the following decomposition of $L(-2\Lambda_0)$ -modules:*

$$\overline{F} \otimes F_{-1} \cong \oplus_{s \in \mathbb{Z}} \pi_s(L(-2\Lambda_0)).$$

Proof. Since $\overline{F} \otimes F_{-1}$ is a vertex superalgebra we have that

$$\text{Ker}_{\overline{F} \otimes F_{-1}} H(0) \cong \mathcal{L}_0(\overline{F}) \cong L(-2\Lambda_0)$$

is a vertex subalgebra of $\overline{F} \otimes F_{-1}$. Now the statement follows from Theorem 8.1. \square

By using Corollary 5.2 and the proof similar to those of Theorem 8.1 one obtains the following result.

Theorem 8.2. *For every $\lambda \in \{-1\} \cup \mathbb{C} \setminus \mathbb{Z}$, the vector space $\tilde{F}_{-\frac{\lambda}{z}} \otimes F_{-1}$ carries an $\hat{\mathfrak{g}}$ -structure uniquely determined by :*

$$\begin{aligned} e(m) &= - \sum_{i \in \mathbb{Z}} i \Psi^+(i - \frac{1}{2}) \otimes e_{m-i-1}^\beta \\ f(m) &= - \sum_{i \in \mathbb{Z}} (i + \lambda) \Psi^-(i - \frac{1}{2}) \otimes e_{m-i-1}^{-\beta} \\ h(m) &= -2\beta(m) + \lambda \delta_{m,0} \\ c &= -2, \end{aligned}$$

where $m \in \mathbb{Z}$. Moreover, $\tilde{F}_{-\frac{\lambda}{z}} \otimes F_{-1}$ is a completely reducible $\hat{\mathfrak{g}}$ -module and

$$\tilde{F}_{-\frac{\lambda}{z}} \otimes F_{-1} \cong \bigoplus_{s \in \mathbb{Z}} \pi_s(L(-(2+\lambda)\Lambda_0 + \lambda\Lambda_1)).$$

9. REALIZATION OF IRREDUCIBLE MODULES ON THE VERTEX ALGEBRA $\Pi(0)$

So far we studied the irreducible $\hat{\mathfrak{g}}$ -modules realized on the Weyl vertex algebra. In this section we shall see that there exists a family of irreducible $\hat{\mathfrak{g}}$ -modules realized on a larger vector space. In order to construct new irreducible representations, we shall study the vertex algebra $\Pi(0) = \mathcal{L}_0(F)$ which contains the Weyl vertex algebra W as a subalgebra.

We shall also identify the irreducible $\hat{\mathfrak{g}}$ -module which are $\mathbb{Z}_{\geq 0}$ -graded but don't belong to the category \mathcal{O} . These modules are irreducible quotients of relaxed Verma modules studied in [FST].

First we recall that the boson-fermion correspondence gives that the fermionic vertex superalgebra F is isomorphic to the lattice vertex superalgebra $V_{\mathbb{Z}}$. Therefore,

$$F \otimes F_{-1} \cong V_L = M(1) \otimes \mathbb{C}[L],$$

where the lattice V_L is the lattice vertex superalgebra associated to the lattice

$$L = \mathbb{Z}\alpha + \mathbb{Z}\beta, \quad \langle \alpha, \alpha \rangle = -\langle \beta, \beta \rangle = 1, \quad \langle \alpha, \beta \rangle = 0.$$

(As usual, $M(1)$ is a level one irreducible module for the Heisenberg algebra $\hat{\mathfrak{h}}_{\mathbb{Z}}$ associated to the abelian algebra $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$ and $\mathbb{C}[L]$ is the group algebra with generators e^{α} and e^{β} .)

The operator H from (6.25) coincides with $\alpha(-1) + \beta(-1)$.

We conclude that the vertex algebra $\Pi(0) = \text{Ker}_{F \otimes F_{-1}}(\alpha + \beta)(0) = \mathcal{L}_0(F)$ is isomorphic to the vertex algebra

$$(9.40) \quad \Pi(0) \cong M(1) \otimes \mathbb{C}[\mathbb{Z}(\alpha + \beta)].$$

Remark 9.1. *By using different methods, the vertex algebra $\Pi(0)$ was also studied by E. Frenkel in [F] and by S. Berman, C. Dong and S. Tan in [BDT].*

Now we shall apply the results from Sections 5 and 6. The following result gives a construction of a large class of irreducible $\hat{\mathfrak{g}}$ -modules on the vertex algebra $\Pi(0)$.

Combining Proposition 5.1 and Corollary 6.1 we obtain the following result.

Theorem 9.1. *Assume that $\lambda \in \mathbb{C} \setminus \mathbb{Z}$ and that*

$$\chi(z) = \sum_{n=-p}^{\infty} \chi_{-n} z^{n-1} \in \mathbb{C}((z)) \quad (p \in \mathbb{Z}_{\geq 0})$$

satisfies the conditions (5.10) and (5.11) of Proposition 5.1. Then on the vertex algebra $\Pi(0) = \mathcal{L}_0(F)$ exists the structure of an irreducible $\hat{\mathfrak{g}}$ -module isomorphic to $\mathcal{L}_0(F(\frac{\lambda}{z}, \chi))$.

Define the following Virasoro vector in $\Pi(0) \subset F \otimes F_{-1}$:

$$\omega = \omega^{(f)} \otimes \mathbf{1} - \frac{1}{2} \mathbf{1} \otimes \beta(-1)^2 = \frac{1}{2}(\alpha(-1)^2 - \beta(-1)^2) \cdot (\mathbf{1} \otimes \mathbf{1}).$$

Let $L(z) = Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2}$. Then the components of the field $L(z)$ satisfies the commutation relations for the Virasoro algebra with central charge $c = 2$. Moreover, $L(0)$ acts semisimply on $F \otimes F_{-1}$ with half-integer eigenvalues, and it defines a $\mathbb{Z}_{\geq 0}$ -graduation on the vertex algebra $\Pi(0)$:

$$(9.41) \quad \Pi(0) = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \Pi(0)_m, \quad \Pi(0)_m = \{v \in \Pi(0) \mid L(0)v = mv\}.$$

Let $\text{ch}_{\Pi(0)}(q, z) = \text{tr } q^{L(0)} z^{-2\beta(0)}$.

By using relation (9.40) and the properties of the δ -function one can easily show the following result.

Proposition 9.1. *We have:*

$$(9.42) \quad \begin{aligned} \text{ch}_{\Pi(0)}(q, z) &= \delta(z^2) \prod_{n=1}^{\infty} (1 - q^n)^{-2} \\ &= \delta(z^2) \prod_{n=1}^{\infty} (1 - q^n z^2)^{-1} (1 - q^n z^{-2})^{-1}. \end{aligned}$$

We are now interested in $\hat{\mathfrak{g}}$ -modules from Theorem 9.1 such that the $\hat{\mathfrak{g}}$ -action is compatible with the $L(0)$ -graduation on the vertex algebra $\Pi(0)$. But the action is compatible with the graduation if and only if $\chi(z) = \frac{\mu}{z}$ for certain $\mu \in \mathbb{C}$. Therefore we should consider the irreducible \mathcal{V} -module $F(\frac{\lambda}{z}, \frac{\mu}{z})$, where $\lambda, \mu \in \mathbb{C} \setminus \mathbb{Z}$. The module $F(\frac{\lambda}{z}, \frac{\mu}{z})$ has the simple structure as an \mathcal{A} -module. In fact, the action of the Lie superalgebra \mathcal{A} is (up to scalar factor) the same as the action of the Clifford algebra CL on F . When we apply Corollary 6.1, we get a family of irreducible $\hat{\mathfrak{g}}$ -modules $\mathcal{L}_s(F(\frac{\lambda}{z}, \frac{\mu}{z}))$ at the critical level.

Now we want to identify these irreducible $\hat{\mathfrak{g}}$ -modules.

For every $s \in \mathbb{Z}$, we define a family of vectors $w_j^{(s)} \in \mathcal{L}_s(F(\frac{\lambda}{z}, \frac{\mu}{z}))$, by

$$\begin{aligned} w_0^{(s)} &:= \mathbf{1} \otimes e^{-s\beta}, \\ w_j^{(s)} &:= \Psi^+(-j + \frac{1}{2}) \cdots \Psi^+(-\frac{1}{2}) \mathbf{1} \otimes e^{(j-s)\beta} \quad (j \in \mathbb{Z}_{>0}), \\ w_{-j}^{(s)} &:= \Psi^-(-j + \frac{1}{2}) \cdots \Psi^-(-\frac{1}{2}) \mathbf{1} \otimes e^{-(j+s)\beta} \quad (j \in \mathbb{Z}_{>0}). \end{aligned}$$

By using a direct calculation, one can prove the following lemma.

Lemma 9.1. *Assume that $s, j \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$. Then we have*

$$\begin{aligned} e(n-s).w_j^{(s)} &= \delta_{n,0}(\lambda + j)w_{j+1}^{(s)}, \\ h(n).w_j^{(s)} &= \delta_{n,0}(2j - 2s + \lambda - \mu)w_j^{(s)}, \\ f(n+s).w_j^{(s)} &= \delta_{n,0}(\mu - j)w_{j-1}^{(s)}. \end{aligned}$$

Let us first consider the case $s = 0$. Define:

$$E_{\lambda, \mu} := \mathcal{L}_0(F(\frac{\lambda}{z}, \frac{\mu}{z})).$$

By construction, $E_{\lambda,\mu} \cong \Pi(0)$ as a vector space. Introduce the graduation operator $L(0)$ on the $\hat{\mathfrak{g}}$ -module $E_{\lambda,\mu}$ by using the vertex algebra graduation (9.41) on $\Pi(0)$. So let

$$E_{\lambda,\mu} = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} E_{\lambda,\mu}(m), \quad E_{\lambda,\mu}(m) = \{v \in E_{\lambda,\mu} \mid L(0)v = mv\}.$$

Since $[L(0), x(n)] = -nx(n)$ for every $x \in \mathfrak{g}$, we have that $\hat{\mathfrak{g}}$ -action on $E_{\lambda,\mu}$ is compatible with the graduation. In other words, $E_{\lambda,\mu}$ is an $\hat{\mathfrak{g}} \oplus \mathbb{C}L(0)$ -module. Lemma 9.1 shows that the top level

$$E_{\lambda,\mu}(0) = \text{span}_{\mathbb{C}}\{w_j^{(0)} \mid j \in \mathbb{Z}\}$$

is an irreducible $U(\mathfrak{g})$ -module which is neither highest nor lowest weight with respect to \mathfrak{g} .

Next we consider general case. By using Lemma 9.1 we have that $\pi_s(\mathcal{L}_s(F(\frac{\lambda}{z}, \frac{\mu}{z})))$ is an irreducible $\mathbb{Z}_{\geq 0}$ -graded $\hat{\mathfrak{g}}$ -module whose top level is isomorphic to $E_{\lambda,\mu}(0)$. This proves that $\pi_s(\mathcal{L}_s(F(\frac{\lambda}{z}, \frac{\mu}{z}))) \cong E_{\lambda,\mu}$. Therefore,

$$\mathcal{L}_s(F(\frac{\lambda}{z}, \frac{\mu}{z})) \cong \pi_{-s}(E_{\lambda,\mu}).$$

In this way we have proved the following result.

Theorem 9.2. *For every $\lambda, \mu \in \mathbb{C} \setminus \mathbb{Z}$, the vector space $F(\frac{\lambda}{z}, \frac{\mu}{z}) \otimes F_{-1} \cong F \otimes F_{-1}$ carries an $\hat{\mathfrak{g}}$ -structure uniquely determined by :*

$$\begin{aligned} e(m) &= \sum_{i \in \mathbb{Z}} (\lambda - i) \Psi^+(i - \frac{1}{2}) \otimes e_{m-i-1}^{\beta} \\ f(m) &= \sum_{i \in \mathbb{Z}} (\mu - i) \Psi^-(i - \frac{1}{2}) \otimes e_{m-i-1}^{-\beta} \\ h(m) &= -2\beta(m) + (\lambda - \mu) \delta_{m,0} \\ c &= -2, \end{aligned}$$

where $m \in \mathbb{Z}$. Moreover, $F(\frac{\lambda}{z}, \frac{\mu}{z}) \otimes F_{-1}$ is a completely reducible $\hat{\mathfrak{g}}$ -module and

$$F(\frac{\lambda}{z}, \frac{\mu}{z}) \otimes F_{-1} \cong \bigoplus_{s \in \mathbb{Z}} \pi_s(E_{\lambda,\mu}).$$

Although modules $E_{\lambda,\mu}$ don't belong to the category \mathcal{O} , one can investigate their characters. The operators $h(0)$ and $L(0)$ acts semisimply on $E_{\lambda,\mu}$ with finite-dimensional common eigenspaces. Since we need the degree operator $L(0)$ we shall consider $E_{\lambda,\mu}$ as a module for the Lie algebra $\hat{\mathfrak{g}} \oplus \mathbb{C}L(0)$.

Corollary 9.1. *For every $\lambda, \mu \in \mathbb{C} \setminus \mathbb{Z}$, the vertex algebra $\Pi(0)$ carries the structure of an irreducible $\mathbb{Z}_{\geq 0}$ -graded $\hat{\mathfrak{g}} \oplus \mathbb{C}L(0)$ -module isomorphic to $E_{\lambda,\mu}$. We have the following character formulae:*

$$\begin{aligned} \text{ch}_{E_{\lambda,\mu}}(q, z) &= \text{tr } q^{L(0)} z^{h(0)} = z^{\lambda-\mu} \delta(z^2) \prod_{n=1}^{\infty} (1 - q^n z^2)^{-1} (1 - q^n z^{-2})^{-1} \\ &= z^{\lambda-\mu} \delta(z^2) \prod_{n=1}^{\infty} (1 - q^n)^{-2}. \end{aligned}$$

Proof. We already saw that $E_{\lambda,\mu}$ is an irreducible $\mathbb{Z}_{\geq 0}$ -graded $\hat{\mathfrak{g}} \oplus \mathbb{C}L(0)$ -module. By using (9.42) we get

$$\mathrm{ch}_{E_{\lambda,\mu}}(q, z) = \mathrm{tr} \, q^{L(0)} z^{h(0)} = z^{\lambda-\mu} \mathrm{ch}_{\Pi(0)}(q, z) = z^{\lambda-\mu} \delta(z^2) \prod_{n=1}^{\infty} (1 - q^n)^{-2}.$$

This proves the character formula. \square

Remark 9.2. *Modules $E_{\lambda,\mu}$ are irreducible quotients of certain relaxed Verma modules which are introduced and studied in [FST]. In our terminology, the relaxed Verma modules are $N(-2, E_{\lambda,\mu}(0))$ and they have the following character*

$$\mathrm{ch}_{N(-2, E_{\lambda,\mu}(0))}(q, z) = z^{\lambda-\mu} \delta(z^2) \prod_{n=1}^{\infty} (1 - q^n)^{-3}.$$

In Corollary 9.1 we calculate the characters of irreducible quotients of relaxed Verma modules by using our explicit realization. So our methods don't use structure theory of relaxed Verma modules.

REFERENCES

- [A1] D. Adamović, Representations of the $N = 2$ superconformal vertex algebra, Internat. Math. Res. Notices **2** (1999) 61-79.
- [A2] D. Adamović, Representations of the vertex algebra $\mathcal{W}_{1+\infty}$ with a negative integer central charge, Comm. Algebra **29** (2001) no. 7, 3153-3166.
- [A3] D. Adamović, A construction of admissible $A_1^{(1)}$ -modules of level $-\frac{4}{3}$, J. Pure Appl. Algebra **196** (2005) 119-134.
- [AM] D. Adamović and A. Milas, Vertex operator algebras associated to the modular invariant representations for $A_1^{(1)}$, Math. Res. Lett. **2** (1995) 563-575.
- [BDT] S. Berman, C. Dong and S. Tan, Representations of a class of lattice type vertex algebras, J. Pure Appl. Algebra **176** (2002) 27-47.
- [DL] C. Dong and J. Lepowsky, Generalized vertex algebras and relative vertex operators, Birkhäuser, Boston, 1993.
- [DM] C. Dong and G. Mason, On quantum Galois theory, Duke Math. J. **86** (1997) 305-321.
- [EG] W. Eholzer and M. R. Gaberdiel, Unitarity of rational $N = 2$ superconformal theories, Comm. Math. Phys. **186** (1997) 61-85.
- [FB] E. Frenkel and D. Ben-Zvi, Vertex algebras and algebraic curves, Mathematical Surveys and Monographs; no. 88, AMS, 2001.
- [F] E. Frenkel, Lectures on Wakimoto modules, opers and the center at the critical level, Adv. Math. **195** (2005) 297-404.
- [FF] E. Frenkel and B. Feigin, Affine Kac-Moody algebras and semi-infinite flag manifolds, Comm. Math. Phys. **128** (1990) 161-189.
- [FHL] I. B. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, Mem. Amer. Math. Soc. **104**, 1993.
- [FLM] I. B. Frenkel, J. Lepowsky and A. Meurman, Vertex Operator Algebras and the Monster, Pure and Applied Math., Vol. **134**, Academic Press, New York, 1988.
- [FMS] D. Friedan, E. Martinec and S. Shenker, Conformal invariance, supersymmetry and string theory, Nuclear Phys. B **271** (1986) 93-165.
- [FST] B. L. Feigin, A. M. Semikhatov and I. Yu. Tipunin, Equivalence between chain categories of representations of affine $sl(2)$ and $N = 2$ superconformal algebras, J. Math. Phys. **39** (1998), 3865-390.
- [FZ] I. B. Frenkel and Y. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras, Duke Math. J. **66** (1992), 123-168.
- [K1] V. Kac, Infinite dimensional Lie algebras, Third edition, Cambridge Univ. Press, Cambridge, 1990.

- [K2] V. Kac, Vertex Algebras for Beginners, University Lecture Series, Second Edition, Amer. Math. Soc., 1998, Vol. 10.
- [KK] V. Kac and D. Kazhdan, Structure of representations with highest weight of infinite dimensional Lie algebras, Adv. Math. **34** (1979) 97-108.
- [KS] Y. Kazama and H. Suzuki, New N=2 superconformal field theories and superstring compactifications, Nucl. Phys. B **321** (1989), 232-268.
- [Li1] H. Li, Local systems of vertex operators, vertex superalgebras and modules, J. Pure Appl. Algebra **109** (1996), 143-195.
- [Li2] H. Li, The physical superselection principle in vertex operator algebra theory, J. Algebra **196** (1997) 436-457.
- [LL] J. Lepowsky and H. Li, Introduction to vertex operator algebras and their representations, Progress in Math., Vol. 227, Birkhäuser, Boston, 2004.
- [MP] A. Meurman and M. Primc, Annihilating fields of standard modules of $\tilde{sl}(2, \mathbb{C})$ and combinatorial identities, Mem. Amer. Math. Soc. **652** 1999.
- [S] M. Szczesny, Wakimoto modules for twisted affine Lie algebras, Math. Res. Lett. **9** (2002), no.4, 433-448.
- [W] M. Wakimoto, Fock representations of affine Lie algebra $A_1^{(1)}$, Comm. Math. Phys. **104** (1986) 605-609.
- [Z] Y. Zhu, Vertex operator algebras, elliptic functions and modular forms, Ph. D. thesis, Yale University, 1990; Modular invariance of characters of vertex operator algebras, J. Amer. Math. Soc. **9** (1996), 237-302.

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